

An algorithm to obtain global solutions of the double confluent Heun equation

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Abstract. A procedure is proposed to construct solutions of the double confluent Heun equation with a determinate behaviour at the singular points. The connection factors are expressed as quotients of Wronskians of the involved solutions. Asymptotic expansions are used in the computation of those Wronskians. The feasibility of the method is shown in an example, namely, the Schrödinger equation with a quasi-exactly-solvable potential

Keywords: differential equations, double confluent Heun equation, connection problem, Stokes phenomenon, multiplicative solutions, asymptotic solutions, Schrödinger equation, quasi-exactly-solvable potentials

Abbreviations: FDE – Fuchsian differential equations; DCHE – double confluent Heun equation

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In memoriam Prof. Luigi Gatteschi

1. Introduction

Fuchsian differential equations (FDE), i. e., homogeneous differential equations with single-valued analytic coefficient functions, play a relevant role in many areas of Physics, Engineering and Statistics. A large class of the special functions appearing in these areas are but solutions of second order FDE. The most studied of these equations is the Riemann hypergeometric equation, that, in its general form, presents three regular singular points. According with the location of its singularities, the equation is known with different names (generalized Legendre equation, Gauss equation, ...). Confluence of two or all three of its singularities transforms the hypergeometric equation in others, with particular names (Kummer, Bessel, Weber, ...), encountered in the solution of a lot of problems.

The next, in order of complexity, FDE is the Heun equation, that presents four regular singularities. Different kinds of confluence of these singularities produce the confluent, double confluent, biconfluent and triconfluent Heun equations. The book edited by Ronveaux (1995) con-



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Table I. Singularities of the different types of Heun equations

equation	rank 0	rank 1	rank 2	rank 3
Heun	0, 1, a (arbitrary $\neq 0, 1, \infty$), ∞			
confluent H.	0, 1	∞		
double confluent H.		0, ∞		
biconfluent H.	0		∞	
triconfluent H.				∞

tains a thorough revision of the results and open problems concerning those equations. More recently, Slavyanov and Lay (2000) have considered again the class of Heun equations, in their study of Special Functions as solutions of differential equations with singular points. Confluence reduces the number of singularities, but increases the rank of some ones. Trivial changes of variable allow to locate the singularities at chosen points. We report in Table I the type of singularities and their positions when those equations are written in their usual conventional form .

In almost all nontrivial applications, in which special functions arise, the problem at hand reduces to a boundary value one on an interval of the real axis whose interior points are ordinary ones, whereas one or both of its ends may be singular points of the differential equation. Usually, all but one of the parameters appearing in the coefficient functions of the equation are fixed. Then, the boundary conditions restrict (quantize) the possible values of the free parameter. In the neighbourhood of each end, a pair of independent solutions with a well known behaviour can be chosen. Such pair can serve as a basis to write, by linear combination, any other solution of the differential equation, in particular, that satisfying the boundary condition at the corresponding end. In order to examine the behaviour of this solution at the other end, one has to be able to write each one of the two solutions forming the basis at one end as linear combination of the basic solutions at the other end. In other words, one has to solve the so called “connection problem” for the ends. Articles by Kohno (1974), Naundorf (1976), Schäfke and Schmidt (1980), and Schäfke (1980, 1984), among others, deal with this problem.

In a recent paper (Gómez and Sesma, 2007) we have suggested a different procedure to solve the connection problem for the cases of one of the ends being an ordinary or a regular singular point, and the other an irregular one. Examples of such cases are found in the triconfluent and biconfluent Heun equations, respectively. Our method

is mostly inspired by the work of Naundorf, but presents some considerable advantages, from the numerical point of view. Our use of the Wronskians of the basic solutions to calculate the connection factors allows us to avoid having to sum slowly convergent double series like those encountered in the Naundorf's procedure. The purpose of this paper is to extend our method to the solution of the connection problem for two irregular singular points, the DCHE being a typical example of this case.

We start by presenting, in the next Section, the DCHE and its basic solutions in the vicinity of its singular points. Our procedure to solve the connection problem and the corresponding algorithms are reported in Sections 3 and 4, respectively. An example, in Section 5, illustrates the method. Section 6 explains how to test the results of the only nontrivial algorithm. Finally, Section 7 contains a comment about the generality of the procedure.

2. The double confluent Heun Equation

A complete discussion of the different forms adopted by the DCHE, under appropriate transformations, has been done by Schmidt and Wolf (1995). We concentrate on what they call *normal form* (Eq. 1.1.10 of their article)

$$D^2 y + B(z) y = 0, \quad D = z \frac{d}{dz}, \quad B(z) = \sum_{p=-2}^2 B_p z^p, \quad (1)$$

that, with the change of dependent variable

$$w(z) = z^{1/2} y(z) \quad (2)$$

and the notation

$$A_0 = B_0 + 1/4, \quad A_p = B_p, \quad p \neq 0, \quad (3)$$

can be written in the form

$$z^2 \frac{d^2 w}{dz^2} + \sum_{p=-2}^2 A_p z^p w = 0, \quad (4)$$

free of first order derivatives and, therefore, preferable for our discussion because of the fact that the Wronskians of their solutions are independent of z . In what follows, we will assume that we are dealing with a *non-degenerate case* of the DCHE, that is, that

$$A_2 A_{-2} \neq 0.$$

The origin and the infinity are the only singularities, both of rank 1, of the differential equation. We are interested in considering three pairs of independent solutions, namely:

- Two *multiplicative* solutions (Arscott, 1987), w_1 and w_2 , that, except for particular sets of values of the parameters A_p in Eq. (4), have the form

$$w_j = z^{\nu_j} \sum_{n=-\infty}^{\infty} c_{n,j} z^n, \quad \text{being} \quad \sum_{n=-\infty}^{\infty} |c_{n,j}|^2 < \infty, \quad j = 1, 2. \quad (5)$$

The indices ν_j are not completely defined. They admit addition of any integer (with an adequate relabeling of the coefficients). To avoid ambiguities, we assume that

$$|\Re \nu_j| \leq 1/2.$$

- Two formal solutions, w_3 and w_4 , that have the nature of asymptotic expansions for $z \rightarrow \infty$,

$$w_k(z) \equiv \exp(\alpha_k z) z^{\mu_k} \sum_{m=0}^{\infty} a_{m,k} z^{-m}, \quad a_{0,k} \neq 0, \quad k = 3, 4. \quad (6)$$

It is usual to say that these two expansions are associated to each other.

- Two formal solutions, w_5 and w_6 , asymptotic expansions for $z \rightarrow 0$, of the form

$$w_l(z) \equiv \exp(\beta_l z^{-1}) z^{\rho_l} \sum_{m=0}^{\infty} b_{m,l} z^m, \quad b_{0,l} \neq 0, \quad l = 5, 6. \quad (7)$$

Also these expansions are associated.

The determinaton of the index ν and the coefficients c_n of the multiplicative solutions is rather laborious. By substitution of (5) in (4) one obtains the infinite set of homogeneous equations for the coefficients

$$(n+\nu)(n-1+\nu) c_n + \sum_{p=-2}^2 A_p c_{n-p} = 0, \quad n = \dots, -1, 0, 1, \dots, \quad (8)$$

that can be interpreted as a nonlinear eigenvalue problem, where the eigenvalue ν must be such that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty. \quad (9)$$

In Section 4 we recall the Newton iterative method to solve that problem. In general, two indices, ν_1 and ν_2 , and two corresponding sets of coefficients, $\{c_{n,1}\}$ and $\{c_{n,2}\}$ are obtained, but for certain sets of values of the parameters A_p only one multiplicative solution appears. Any other independent solution must include powers of the variable multiplied by its logarithm. Such logarithmic solutions cannot correspond, usually, to the practical system that one tries to describe and are, therefore, to be discarded. We will assume, from now on, that the parameters A_p are such that the DCHE admits two independent multiplicative solutions. In what concerns the formal solutions, the exponents α , μ , β , and ρ and the coefficients a_m and b_m of the expansions (6) and (7) must be such that these expansions satisfy the differential equation. One obtains in this way

$$\alpha_k = \sqrt{-A_2}, \quad \mu_k = -A_1/2\alpha_k, \quad (10)$$

$$\beta_l = \sqrt{-A_{-2}}, \quad \rho_l = 1 + A_{-1}/2\beta_l, \quad (11)$$

for the exponents, and the recurrence relations

$$2\alpha_k m a_{m,k} = (m-\mu_k)(m-1-\mu_k) a_{m-1,k} + \sum_{p=0}^2 A_{-p} a_{m-1-p,k}, \quad (12)$$

$$2\beta_l m b_{m,l} = (m-1+\rho_l)(m-2+\rho_l) b_{m-1,l} + \sum_{p=0}^2 A_p b_{m-1-p,l}, \quad (13)$$

for the coefficients. The two independent solutions labeled by $k = 3, 4$ and those by $l = 5, 6$ correspond to the two opposite values of the square root function in (10) and (11).

3. The connection factors

Any solution w of the DCHE can be written as a linear combination of the two multiplicative solutions,

$$w = \zeta_1 w_1 + \zeta_2 w_2. \quad (14)$$

Its behavior in the neighbourhood of the singular points can be immediately written if, besides the coefficients ζ_1 and ζ_2 , one knows the behaviour of the multiplicative solutions, that is, if one knows the *connection factors* T of their asymptotic expansions,

$$w_j \sim T_{j,3} w_3 + T_{j,4} w_4, \quad \text{for } z \rightarrow \infty, \quad j = 1, 2, \quad (15)$$

$$w_j \sim T_{j,5} w_5 + T_{j,6} w_6, \quad \text{for } z \rightarrow 0, \quad j = 1, 2. \quad (16)$$

These connection factors are, obviously, numerical constants, but their values depend on the sector of the complex plane where z lies. This fact, known as “Stokes phenomenon” (Dingle, 1973), introduces a slight complication in the procedure. As it is well known, the connection factor multiplying each one of the asymptotic expansions in the right hand sides of (15) and (16) takes different values in the sectors of the complex z -plane separated by a Stokes ray of the associated expansion. On the ray, the value of the connection factor is the average of those two different ones. In the DCHE, both singular points have rank 1 and, consequently, for each one of the expansions (6) and (7), the sectors delimited by two contiguous Stokes rays of the same expansion have amplitude 2π . In fact, the Stokes rays for those expansions are given by

$$\arg z = -\arg \alpha_k \pm 2n\pi, \quad n = 0, 1, 2, \dots, \quad \text{for } w_k, \quad k = 3, 4, \quad (17)$$

$$\arg z = \arg \beta_l \pm 2n\pi, \quad n = 0, 1, 2, \dots, \quad \text{for } w_l, \quad l = 5, 6 \quad (18)$$

To be specific, let us assume that the labels $\{3, 4\}$ and $\{5, 6\}$ are assigned in such a manner that

$$\text{either } -\pi \leq \arg \alpha_3 < -\pi/2 \quad \text{or} \quad \pi/2 \leq \arg \alpha_3 < \pi, \quad (19)$$

$$-\pi/2 \leq \arg \alpha_4 < \pi/2, \quad (20)$$

$$\text{either } -\pi < \arg \beta_5 \leq -\pi/2 \quad \text{or} \quad \pi/2 < \arg \beta_5 \leq \pi, \quad (21)$$

$$-\pi/2 < \arg \beta_6 \leq \pi/2. \quad (22)$$

Then in the principal Riemann sheet, $-\pi < \arg z \leq \pi$, $T_{j,3}$ and $T_{j,4}$ change their values as z crosses respectively the rays $\arg z = -\arg \alpha_4$ and $\arg z = -\arg \alpha_3$, whereas $T_{j,5}$ and $T_{j,6}$ change at $\arg z = \arg \beta_6$ and $\arg z = \arg \beta_5$, respectively.

Now, we are going to present our procedure to calculate the connection factors. Let $\{r, s\}$ denote either the pair $\{3, 4\}$ or $\{5, 6\}$. From Eqs. (15) or (16), it is immediate to obtain

$$T_{j,r} = \frac{\mathcal{W}[w_j, w_s]}{\mathcal{W}[w_r, w_s]}, \quad T_{j,s} = \frac{\mathcal{W}[w_j, w_r]}{\mathcal{W}[w_s, w_r]}, \quad (23)$$

where $\mathcal{W}[f, g]$ represents the Wronskian of the functions f and g ,

$$\mathcal{W}[f, g](z) \equiv f(z) g'(z) - f'(z) g(z). \quad (24)$$

There is no difficulty in using the asymptotic expansions (6) and (7) to calculate the denominators in (23). In fact,

$$\mathcal{W}[w_3, w_4] = -\mathcal{W}[w_4, w_3] = 2\alpha_4 a_{0,3} a_{0,4}, \quad (25)$$

$$\mathcal{W}[w_5, w_6] = -\mathcal{W}[w_6, w_5] = 2\beta_5 b_{0,5} b_{0,6}. \quad (26)$$

The numerators, instead, require a more sophisticated method. Direct evaluation would give asymptotic expansions, containing both positive and negative powers of the variable, which result to be inadequate to obtain precise values of those numerators. But we can benefit from the fact that, whenever one considers not merely a ray, but a sector of the complex plane, an asymptotic power series defines unambiguously an analytic function. Our idea is to write, for each one of the connection factors $T_{j,t}$ ($t = r, s$), two asymptotic expansions of the same function, one of the expansions containing, as a common multiplicative constant, the Wronskian in the numerator of the expression of $T_{j,t}$. Comparison of coefficients of equal powers of the variable in the two expansions gives immediately that numerator. With that purpose, we introduce some auxiliary functions and asymptotic expansions, namely

$$u_j(z) \equiv e^{-\alpha_t z/2} w_j, \quad u_t(z) \equiv e^{-\alpha_t z/2} w_t, \quad j = 1, 2, \quad t = 3, 4, \quad (27)$$

$$v_j(z) \equiv e^{-\beta_t z/2} w_j, \quad v_t(z) \equiv e^{-\beta_t z/2} w_t, \quad j = 1, 2, \quad t = 5, 6, \quad (28)$$

whose Wronskians obey the relations

$$\mathcal{W}[u_j, u_t] = e^{-\alpha_t z} \mathcal{W}[w_j, w_t], \quad j = 1, 2, \quad t = 3, 4, \quad (29)$$

$$\mathcal{W}[v_j, v_t] = e^{-\beta_t z} \mathcal{W}[w_j, w_t], \quad j = 1, 2, \quad t = 5, 6. \quad (30)$$

Direct calculation of the left hand sides of these equations gives the expansions

$$\mathcal{W}[u_j, u_t] = \sum_{n=-\infty}^{\infty} \gamma_{n,j,t} z^{n+\nu_j+\mu_t}, \quad j = 1, 2, \quad t = 3, 4, \quad (31)$$

$$\mathcal{W}[v_j, v_t] = \sum_{n=-\infty}^{\infty} \eta_{n,j,t} z^{n+\nu_j+\rho_t}, \quad j = 1, 2, \quad t = 5, 6, \quad (32)$$

with coefficients

$$\gamma_{n,j,t} = \sum_{m=0}^{\infty} a_{m,t} [\alpha_t c_{n+m,j} - (n+2m+1+\nu_j-\mu_t)c_{n+m+1,j}], \quad (33)$$

$$\eta_{n,j,t} = \sum_{m=0}^{\infty} b_{m,t} [-\beta_t c_{n-m+2,j} - (n-2m+1+\nu_j-\rho_t)c_{n-m+1,j}], \quad (34)$$

Now, in order to compare coefficients of equal powers in the two sides, we need to write expansions of the right hand sides of (29) and (30) in powers of z with the same exponents as in the left hand sides. This can be done by using the Heaviside's exponential series (Hardy, 1949)

$$e^{\xi} \sim \sum_{n=-\infty}^{\infty} \frac{\xi^{n+\delta}}{\Gamma(n+1+\delta)}, \quad |\arg \xi| < \pi, \quad \delta \text{ arbitrary}, \quad (35)$$

with ξ replaced respectively by $-\alpha_t z$ and $-\beta_t z^{-1}$, and taking δ equal to $\nu_j + \mu_k$ and $-(\nu_j + \rho_t)$. This kind of representation of the exponential function has already been used by Naundorf in his solution of the connection problem. Notice that the restriction $|\arg \xi| < \pi$ prevents the use of such expansions if $\arg z = -\arg \alpha_t$ or $\arg z = \arg \beta_t$. But these values of $\arg z$ correspond precisely to the rays at which the connection factor to be calculated changes. As we have already said, the value assigned to the connection factor in that situation is the average of its values in the sectors separated by that ray, where the restriction $|\arg \xi| < \pi$ is fulfilled. Following that procedure, one obtains, for z out of the Stokes ray of w_t in the principal Riemann sheet,

$$\mathcal{W}[w_j, w_t] = \frac{\Gamma(n+1+\nu_j+\mu_t)}{(-\alpha_t)^{n+\nu_j+\mu_t}} \gamma_{n,j,t}, \quad j = 1, 2, \quad t = 3, 4, \quad (36)$$

$$\mathcal{W}[w_j, w_t] = \frac{\Gamma(n+1-\nu_j-\rho_t)}{(-\beta_t)^{n-\nu_j-\rho_t}} \eta_{-n,j,t}, \quad j = 1, 2, \quad t = 5, 6, \quad (37)$$

where the minus sign in front of α_t and β_t is to be interpreted as $e^{i\pi}$ or $e^{-i\pi}$ so as to have $|\arg(-\alpha_t z)| < \pi$ and $|\arg(-\beta_t z^{-1})| < \pi$. If z lies on the Stokes ray of w_t , one has, instead of (36) and (37),

$$\mathcal{W}[w_j, w_t] = (-1)^n \cos[\pi(\nu_j + \mu_t)] \frac{\Gamma(n+1+\nu_j+\mu_t)}{\alpha_t^{n+\nu_j+\mu_t}} \gamma_{n,j,t}, \quad (38)$$

$$\mathcal{W}[w_j, w_t] = (-1)^n \cos[\pi(\nu_j + \rho_t)] \frac{\Gamma(n+1-\nu_j-\rho_t)}{\beta_t^{n-\nu_j-\rho_t}} \eta_{-n,j,t}. \quad (39)$$

4. The algorithms

We present in this Section algorithms that facilitate implementation of the procedure sketched above. Its successive steps are considered separately in different subsections

4.1. MULTIPLICATIVE SOLUTIONS

As it has been already said, we face in Eqs. (8) and (9) a nonlinear eigenvalue problem. Algorithms to solve finite order problems of this kind have been discussed by Ruhe (1973). The condition (9) implies

$$\lim_{n \rightarrow \pm\infty} |c_n| = 0, \quad (40)$$

that makes possible to reduce, by truncation, our problem (8) to one with n going from $-M$ to N , both M and N being positive and

sufficiently large to ensure that the solution of the truncated problem approximates that of the original one. Each step of the Newton iteration method consists in moving from an approximate solution, $\{\nu^{(i)}, c_n^{(i)}\}$, to another one, $\{\nu^{(i+1)}, c_n^{(i+1)}\}$, by solving the system of equations

$$\begin{aligned} & \left(2n-1+2\nu^{(i)}\right)c_n^{(i)}\left(\nu^{(i+1)} - \nu^{(i)}\right) + \left(n+\nu^{(i)}\right)\left(n-1+\nu^{(i)}\right)c_n^{(i+1)} \\ & + \sum_{p=-2}^2 A_p c_{n-p}^{(i+1)} = 0, \quad n = -M, \dots, -1, 0, 1, \dots, N, \end{aligned} \quad (41)$$

$$\sum_{n=-M}^N c_n^{(i)*} c_n^{(i+1)} = 1, \quad (42)$$

that results, by linearization, from (8) and from the truncated normalization condition

$$\sum_{n=-M}^N |c_n|^2 = 1.$$

Needless to say, the values of $c_m^{(i)}$ with $m < -M$ or $m > N$ entering in some of the equations (41) should be taken equal to zero, in accordance with the truncation done. As usual, the iteration process is stopped when the difference between consecutive solutions is satisfactory. Then the process is repeated with larger and larger values of M and N , to obtain a stable solution within the required precision.

The iteration process just described needs initial values $\{\nu^{(0)}, c_n^{(0)}\}$ not far from the true solution. The two different values of ν can be obtained from the two eigenvalues

$$\lambda_j = \exp(2i\pi\nu_j) \quad (43)$$

of the circuit matrix \mathbb{C} (Wasow, 2002) for the singular point at $z = 0$. The entries of that matrix can be computed by numerically integrating the DCHE on the unit circle, from $z = \exp(0)$ to $z = \exp(2i\pi)$, for two independent sets of initial values. If we consider two solutions, $w_a(z)$ and $w_b(z)$, obeying, for instance, the conditions

$$\begin{aligned} w_a(e^0) &= 1, & w'_a(e^0) &= 0, \\ w_b(e^0) &= 0, & w'_b(e^0) &= 1, \end{aligned}$$

then

$$\begin{aligned} C_{11} &= w_a(e^{2i\pi}), & C_{12} &= w_b(e^{2i\pi}), \\ C_{21} &= w'_a(e^{2i\pi}), & C_{22} &= w'_b(e^{2i\pi}), \end{aligned}$$

and

$$\nu = \frac{1}{2i\pi} \ln \left[\frac{1}{2} \left(C_{11} + C_{22} \pm \sqrt{(C_{11} - C_{22})^2 + 4C_{12}C_{21}} \right) \right]. \quad (44)$$

The two signs in front of the square root produce two different values for ν , unless the parameters A_p in the DCHE be such that $(C_{11} - C_{22})^2 + 4C_{12}C_{21} = 0$, in which case only one multiplicative solution appears, any other independent solution containing logarithmic terms. The ambiguity in the real part of ν due to the multivaluedness of the logarithm in the right hand side of (44) is eliminated by the restriction $|\Re \nu| \leq 1/2$ assumed above. Notice that

$$\lambda_1 \lambda_2 = \det \mathbb{C} = \mathcal{W}[w_a, w_b] = 1$$

and, therefore,

$$\nu_1 + \nu_2 = 0 \pmod{1}.$$

This may serve as a test for the integration of the DCHE on the unit circle.

Although Eq. (44) is exact, the C_{mn} are obtained numerically and the resulting values of ν may only be considered as starting values, $\nu_j^{(0)}$ ($j = 1, 2$), for the Newton iteration process. As starting coefficients $c_n^{(0)}$ one may use the solutions of the homogeneous system

$$(n + \nu_j^{(0)})(n - 1 + \nu_j^{(0)}) c_{n,j}^{(0)} + \sum_{p=-2}^2 A_p c_{n-p,j}^{(0)} = 0 \\ n = -M, \dots, -1, 0, 1, \dots, N, \quad j = 1, 2, \quad (45)$$

with the already mentioned truncated normalization condition

$$\sum_{n=-M}^N |c_{n,j}^{(0)}|^2 = 1. \quad (46)$$

4.2. FORMAL SOLUTIONS

The exponents of the formal expansions (6) and (7) are given in (10) and (11). The coefficients must obey the recurrence relations (12) and (13) that are but third order difference equations. The Perron-Kreuser theorem (Perron, 1959) predicts for each one of them a unique (save for multiplication by a constant) dominant solution that can be obtained starting, for instance, with

$$a_{0,k} = 1, \quad b_{0,l} = 1, \quad k = 3, 4, \quad l = 5, 6, \quad (47)$$

and using the recurrence relations as they appear in (12) and (13). To avoid overflows, it may be convenient to deal with the quotients of successive coefficients, that satisfy

$$\begin{aligned} 2\alpha_k m \frac{a_{m,k}}{a_{m-1,k}} &= (m-\mu_k)(m-1-\mu_k) + A_0 \\ &\quad + \frac{A_{-1}}{\frac{a_{m-1,k}}{a_{m-2,k}}} + \frac{A_{-2}}{\frac{a_{m-1,k}}{a_{m-2,k}} \frac{a_{m-2,k}}{a_{m-3,k}}}, \end{aligned} \quad (48)$$

$$\begin{aligned} 2\beta_l m \frac{b_{m,l}}{b_{m-1,l}} &= (m-1+\rho_l)(m-2+\rho_l) + A_0 \\ &\quad + \frac{A_1}{\frac{b_{m-1,l}}{b_{m-2,l}}} + \frac{A_2}{\frac{b_{m-1,l}}{b_{m-2,l}} \frac{b_{m-2,l}}{b_{m-3,l}}}, \end{aligned} \quad (49)$$

4.3. CONNECTION FACTORS

For the computation of the connection factors one should make use of Eqs. (23), (25), (26), (36) and (37), complemented by (33) and (34) conveniently truncated. The integer n in Eqs. (33), (34), (36) and (37) may be chosen at will, with the only restriction of being positive and satisfying

$$(n + \nu)(n + \nu - 1) > \sum_{p=-2}^2 |A_p|. \quad (50)$$

Use of different values of n may serve as a test of the procedure. The sums in (33) and (34), truncated and written in terms of quotients of successive coefficients, $a_{m,k}/a_{m-1,k}$ and $b_{m,l}/b_{m-1,l}$, may be computed in nested form.

5. An example

A particular case of DCHE on the positive real semiaxis, $z \in [0, +\infty)$, is the Schrödinger equation

$$-\frac{\hbar^2}{2m} \left(\frac{d^2 R(r)}{dr^2} - \frac{l(l+1)}{r^2} R(r) \right) + V(r) R(r) = E R(r), \quad (51)$$

for the reduced radial wave function $R(r)$ of a particle of mass m , angular momentum $l\hbar$ and energy $E = A_2\hbar^2/2mr_0^2$ in a spherically symmetric potential

$$V(r) = -\frac{\hbar^2}{2m} \left(\frac{A_{-2} r_0^2}{r^4} + \frac{A_{-1} r_0}{r^3} + \frac{A_0 + l(l+1)}{r^2} + \frac{A_1 r_0^{-1}}{r} \right). \quad (52)$$

In fact, by using the variable z and the wave function w given by

$$z = r/r_0 \quad \text{and} \quad w(z) = R(r), \quad (53)$$

the Schrödinger equation (51) adopts the form (4). The potential (52), with some restrictions on the values of the parameters, belongs to a class of quasi-exactly-solvable ones (Turbiner, 1988). In fact, for the particular set of parameters

$$A_{-2} = -1, \quad A_{-1} = 4/5, \quad A_0 = 31/25, \quad A_1 = 3/5, \quad (54)$$

it presents an $l = 0$ bound state of energy $E = -(1/4)\hbar^2/2mr_0^2$ (Özçelik and Şimşek, 1991). In other words, Eq. (4), with the values of the A_p given by (54) and by $A_2 = -1/4$, has a normalizable solution on the positive real semiaxis. To illustrate the procedure described in the preceding Sections, we have applied it to find global solutions, for $z \in [0, +\infty)$, of (4) with fixed values (54) of the potential parameters and several values

$$A_2 = -1/10, -1/5, -1/4, -3/10, -2/5, \quad (55)$$

of the energy parameter. The results are shown in Tables II to V. We report the output of our double precision FORTRAN codes, but, due to roundoff errors, we do not claim that all the digits reproduced are correct. In fact, entries that in the tables appear as having modulus less than 10^{-12} should be exactly equal to zero. Table II shows the index ν_1 of the multiplicative solution w_1 . Of course, the index ν_2 of w_2 is the opposite, $\nu_2 = -\nu_1$. The connection factors of both multiplicative solutions with the formal ones are listed in Table III. These factors depend on the normalization adopted for the different solutions. We have taken

$$c_{0,j} = 1, \quad j = 1, 2,$$

for the multiplicative solutions, whereas the normalization of the formal solutions is determined by

$$a_{0,k} = 1, \quad k = 3, 4, \quad b_{0,l} = 1, \quad l = 5, 6.$$

Table IV gives the coefficients ζ_1 and ζ_2 of a linear combination of the multiplicative solutions,

$$w_{\text{reg}} = \zeta_1 w_1 + \zeta_2 w_2,$$

well behaved near the origin on the positive real semiaxis, that is, such that

$$w_{\text{reg}}(z) \sim w_5(z) \quad \text{as } z \rightarrow 0^+.$$

Table II. Indices of the multiplicative solutions of the DCHE for the sets of parameters given in (54) and (55). The two indices have opposite values; so the real and imaginary parts of only one of them are shown.

A_2	$\Re \nu_1$	$\Im \nu_1$
-1/10	-.500000000000E+00	-.703150555392E+00
-1/5	-.500000000000E+00	-.531738153810E+00
-1/4	-.400000000000E+00	.102867462041E-26
-3/10	-.262974969075E-15	.509507933497E+00
-2/5	.181198729462E-16	.688682990633E+00

Finally, Table V shows the connection factors giving the behaviour of w_{reg} at infinity on the positive real semiaxis,

$$w_{\text{reg}}(z) \sim T_{\text{reg},3} w_3(z) + T_{\text{reg},4} w_4(z) \quad \text{as } z \rightarrow +\infty.$$

We have already mentioned that the DCHE (4) with parameters as given by (54) and by $A_2 = -1/4$ possesses a solution,

$$w(z) = z^{3/5} \exp(-z^{-1} - z/2), \quad (56)$$

normalizable on the positive real semiaxis. This fact allows to know the exact values of the connection factors $T_{1,3}$ and $T_{1,5}$ of the multiplicative solution

$$w_1(z) = \sum_{n=-\infty}^{\infty} c_{n,1} z^{n-2/5}. \quad (57)$$

Writing (56) in the form

$$w(z) = \sum_{n=-\infty}^{\infty} \hat{c}_n z^{n+3/5}, \quad (58)$$

with

$$\hat{c}_n = \begin{cases} (-1)^n \sum_{m=0}^{\infty} \frac{2^{-m}}{m!(m-n)!} & \text{for } n < 0, \\ (-1)^n \sum_{m=0}^{\infty} \frac{2^{-m-n}}{m!(m+n)!} & \text{for } n \geq 0, \end{cases}$$

comparison with (57) gives immediately

$$T_{1,3} = T_{1,5} = c_{0,1}/\hat{c}_{-1}$$

Table III. Connection factors of the multiplicative solutions, w_1 and w_2 , with the formal solutions at infinity, w_3 and w_4 , and at the origin, w_5 and w_6 , for the DCHE (4) with parameters (54) and (55).

A_2	$\Re T_{1,3}$	$\Im T_{1,3}$	$\Re T_{1,4}$	$\Im T_{1,4}$
-0.1	-.130166356702E+01	.218375080422E+01	.537554017411E+00	.314875723054E+00
-0.2	-.204239771333E+01	.362437608911E+00	.462340626214E-01	.190707221682E+00
-0.25	-.786334477859E+00	.770822392694E-13	.227716282126E-13	.938023374790E-28
-0.3	.107546538975E+00	.137759967302E+01	-.106218738591E+00	.108713582942E-01
-0.4	.108556195947E+01	.199397670471E+01	-.125046997516E+00	.673923636677E-01
A_2	$\Re T_{2,3}$	$\Im T_{2,3}$	$\Re T_{2,4}$	$\Im T_{2,4}$
-0.1	-.179473952568E+00	.141771950373E+01	-.347068293144E+00	-.466342783505E-01
-0.2	.104106884981E+01	.818532573406E+00	-.834077925914E-01	.934810009512E-01
-0.25	.491548162537E+00	.488371898613E-13	-.289740699463E-01	-.293358750439E-14
-0.3	.107546538975E+00	-.137759967302E+01	-.106218738591E+00	-.108713582942E-01
-0.4	.108556195947E+01	-.199397670471E+01	-.125046997516E+00	-.673923636677E-01
A_2	$\Re T_{1,5}$	$\Im T_{1,5}$	$\Re T_{1,6}$	$\Im T_{1,6}$
-0.1	-.350619618820E+01	-.780802661791E+00	.283770014388E-01	-.136520514662E+00
-0.2	-.218278236288E+01	-.183368690581E+00	.349930523633E-02	-.829409172034E-01
-0.25	-.786334477859E+00	-.770774921662E-13	.581790855708E-15	.371252705024E-27
-0.3	-.185903592530E+00	.135138118813E+01	.590575874067E-01	.525872360505E-02
-0.4	-.379273027228E-01	.219921373757E+01	.927424524732E-01	.159396234117E-03
A_2	$\Re T_{2,5}$	$\Im T_{2,5}$	$\Re T_{2,6}$	$\Im T_{2,6}$
-0.1	.182308938715E+01	.867938208517E+00	.346916923655E-01	-.702844924334E-01
-0.2	.128432288745E+01	.553414307656E+00	.229815763899E-01	-.477580284950E-01
-0.25	.493413857062E+00	-.477076860664E-13	.144870349731E-01	-.142012480382E-14
-0.3	-.185903592530E+00	-.135138118813E+01	.590575874067E-01	-.525872360505E-02
-0.4	-.379273027228E-01	-.219921373757E+01	.927424524732E-01	-.159396234117E-03

and, having chosen $c_{0,1} = 1$,

$$T_{1,3} = T_{1,5} = - \left[\sum_{m=0}^{\infty} \frac{2^{-m}}{m!(m+1)!} \right]^{-1}.$$

The values of $\Re T_{1,3}$ and $\Re T_{1,5}$ obtained with our procedure and reported in Table III coincide with the exact values up to all shown digits.

Table IV. Coefficients of the linear combination of multiplicative solutions resulting in a solution $w_{\text{reg}} = \zeta_1 w_1 + \zeta_2 w_2$ normalizable, on the positive real semiaxis, at $z = 0$. The parameters of the DCHE are the same as in Tables II and III.

A_2	$\Re \zeta_1$	$\Im \zeta_1$	$\Re \zeta_2$	$\Im \zeta_2$
-0.1	-.136295750328E+00	.283302821758E-01	.222074702056E+00	-.109613756594E+00
-0.2	-.228256434722E+00	.963021587152E-02	.322449452131E+00	-.155165465358E+00
-0.25	-.127172345631E+01	.124655928872E-12	.510716705831E-13	.325901912099E-25
-0.3	-.325468718708E-01	-.365514500226E+00	-.325468718708E-01	.365514500226E+00
-0.4	-.390741170002E-03	-.227347243108E+00	-.390741170002E-03	.227347243108E+00

Table V. Connection factors of the “regular” solution at the origin with the formal solutions w_3 and w_4 for $z \rightarrow \infty$ along the positive real semiaxis. The parameters in the DCHE are the same as in the preceding Tables.

A_2	$\Re T_{\text{reg},3}$	$\Im T_{\text{reg},3}$	$\Re T_{\text{reg},4}$	$\Im T_{\text{reg},4}$
-0.1	.231089872113E+00	-.485000928307E-12	-.164373692458E+00	.317038062470E-13
-0.2	.925400135830E+00	.898559004980E-12	-.247795480195E-01	.560940183192E-13
-0.25	.100000000000E+01	-.196048546485E-12	-.304389678921E-13	.162523413961E-26
-0.3	.100006470514E+01	.000000000000E+00	.148614535378E-01	.000000000000E+00
-0.4	.905801865774E+00	.000000000000E+00	.307406581930E-01	.000000000000E+00

6. A test of the multiplicative solutions

We have described, in subsection 4.1, an algorithm to obtain the multiplicative solutions, w_1 and w_2 , mentioned in Eq. (5). A test of the correct implementation of the algorithm could be to try to reproduce the particular solution of the DCHE provided by the Maple 11 system. The statement

$$\text{HeunD}(\alpha, \beta, \gamma, \delta, t)$$

gives the value at t ($|t| < 1$) of a function $y(t)$ that obeys the DCHE, written in the Jaffé-Lay form (Slavyanov and Lay, 2000),

$$\frac{d^2y(t)}{dt^2} - \frac{\alpha + 2t + \alpha t^2 - 2t^3}{(t^2 - 1)^2} \frac{dy}{dt} + \frac{\delta + (2\alpha + \gamma)t + \beta t^2}{(t^2 - 1)^3} y = 0, \quad (59)$$

with singular points located at $t = -1$ and $t = 1$, and satisfies the boundary conditions

$$y(0) = 1, \quad y'(0) = 0. \quad (60)$$

(We adopt the notation of the Maple manual. Needless to say, the parameters represented by the symbols α and β along this Section are not related to the exponents α_k , $k = 3, 4$, and β_l , $l = 5, 6$ of the formal solutions (6) and (7).) The changes of independent and dependent variables

$$t = \frac{z-1}{z+1} \quad \text{and} \quad y(t) = z^{-1/2} \exp\left(\frac{\alpha}{8}\left(z - \frac{1}{z}\right)\right) w(z) \quad (61)$$

transform Eq. (59) in Eq. (4) with parameters

$$\begin{aligned} A_{-2} &= -\frac{\alpha^2}{64}, & A_{-1} &= \frac{\gamma - \beta - \delta}{16}, & A_0 &= \frac{8 - \alpha^2 + 4(\beta - \delta)}{32}, \\ A_1 &= \frac{-\gamma - \beta - \delta}{16}, & A_2 &= -\frac{\alpha^2}{64}. \end{aligned} \quad (62)$$

Obviously, $w(z)$ in (61) can be written as a linear combination

$$w(z) = \xi_1 w_1(z) + \xi_2 w_2(z) \quad (63)$$

of the two multiplicative solutions, the coefficients ξ_1 and ξ_2 depending on the parameters α , β , γ and δ . Then, the procedure to check the algorithm used to obtain the multiplicative solutions would run along the following steps:

- Choose a set of values for the parameters α , β , γ and δ .
- Compute the corresponding values of A_p , $p = -2, -1, \dots, 2$, by means of the relations (62).
- Use the algorithm described in Subsection 4.1 to obtain the two multiplicative solutions.
- Determine the coefficients ξ_1 and ξ_2 in (63) by requiring that the second of Eqs. (61) and that resulting by deriving it with respect to t be satisfied for $t = 0$ and $z = 1$. In other words, solve the system

$$w_1(z=1)\xi_1 + w_2(z=1)\xi_2 = 1, \quad (64)$$

$$w'_1(z=1)\xi_1 + w'_2(z=1)\xi_2 = -1/2. \quad (65)$$

- Choose an arbitrary value of t inside the unit circle and compute the corresponding value of z , according to the first of Eqs. (61).
- Check the fulfilment of the second of Eqs. (61) for those values of t and z , the left hand side being computed by the above mentioned Maple statement and the right hand side by using the multiplicative solutions and the coefficients ξ_1 and ξ_2 .

Table VI. Coefficients $c_{n,j}$ of the multiplicative solutions, $w_j(z) = z^{\nu_j} \sum_{n=-\infty}^{\infty} c_{n,j} z^n$, $j = 1, 2$, normalized in such a way that $c_{0,1} = c_{0,2} = 1$. Only the most relevant coefficients are shown. The indices are $\nu_1 = -\nu_2 = 0.346120772343$.

m	$c_{-m,1}$	$c_{m,1}$	$c_{-m,2}$	$c_{m,2}$
1	.167014439267E+00	.190993398030E+01	-.107279310787E+00	.382083714867E+00
2	.481375140054E-01	-.501405710995E-02	.395518603946E-01	.343615502071E+00
3	.256845652795E-02	.651050357942E-01	-.297089364185E-02	.126405802915E-01
4	.670742341737E-03	-.688158578777E-03	.484111513331E-03	.916616846207E-02
5	.151109427662E-04	.719872336102E-03	-.276617916112E-04	.120052304745E-03
6	.436467584453E-05	-.788297814632E-05	.284956022379E-05	.881920361011E-04
7	.424159499646E-07	.392398781019E-05	-.131313020620E-06	.507159044452E-06
8	.163574045519E-07	-.405522033871E-07	.982771078139E-08	.435958940659E-06
9	.537668556535E-10	.127234781325E-07	-.379605967044E-09	.109230687414E-08
10	.397572715764E-10	-.122029907678E-09	.222381579709E-10	.131080043658E-08
11	-.119341203300E-13	.273442588328E-10	-.739683873557E-12	.105746798402E-11
12	.676604265157E-13	-.243166143611E-12	.355327272482E-13	.264897017572E-11
13	-.164661027834E-15	.417999894189E-13	-.103815479335E-14	-.464027978928E-15
14	.850622285001E-16	-.345793838238E-15	.422141078690E-16	.384446086066E-14
15	-.319253539778E-18	.477715538676E-16	-.109995830505E-17	-.299696460688E-17
16	.821887767466E-19	-.369227215875E-18	.387438635716E-19	.420067427815E-17
17	-.376967703210E-21	.423582849020E-19	-.911209342015E-21	-.490269012114E-20
18	.629190033709E-22	-.307215713853E-21	.282929069113E-22	.358011576237E-20
19	-.321864802770E-24	.299866441211E-22	-.606469439157E-24	-.509135489559E-23
20	.390955943808E-25	-.204922402639E-24	.168292193160E-25	.244636867112E-23

Let us choose, for instance, the parameters

$$\alpha = 4, \quad \beta = -3, \quad \gamma = 2, \quad \delta = -1,$$

for which the corresponding parameters in Eq. (4) are

$$A_{-2} = -1/4, \quad A_{-1} = 3/8, \quad A_0 = -1/2, \quad A_1 = 1/8, \quad A_2 = -1/4,$$

The multiplicative solutions, obtained with a double precision FORTRAN code, are then those given in Table VI, and the coefficients in (63) giving the $w(z)$ corresponding to the Maple particular solution turn out to be

$$\xi_1 = -0.48013092979925, \quad \xi_2 = 1.5087428376316.$$

7. Final comment

The procedure presented in the preceding Sections referred to global solutions in the interval between two adjacent irregular singular points, of rank 1, of a second order differential equation like the DCHE. It can be trivially extended, however, to the case of two irregular singular points of arbitrary rank. The only changes due to the larger rank of the singularities are in the evaluation of the Wronskians of the pairs formed by each one of the multiplicative solutions and each one of the formal solutions. The steps are analogous to those detailed in a paper (Gómez and Sesma, 2007) dealing with the connection problem for an ordinary or regular singular point and an irregular one of arbitrary rank. Obviously, the series solutions around the ordinary or regular singular point in that reference should be replaced by the multiplicative solutions of the problem at hand. Then, having computed the connection factors of each multiplicative solution with the formal solutions at each singular point, the procedure is the same as described above.

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